

Existence and stability of global large strong solutions for the Hall-MHD system

Maicon J. Benvenuti* & Lucas C. F. Ferreira†

Abstract

We consider the 3D incompressible Hall-MHD system and prove a stability theorem for global large solutions under a suitable integrable hypothesis in which one of the parcels is linked to the Hall term. As a byproduct, a class of global strong solutions is obtained with large velocities and small initial magnetic fields. Moreover, we prove the local-in-time well-posedness of H^2 -strong solutions which improves previous regularity conditions on initial data.

AMS 2010 MSC: 35Q35, 76D03, 35B35, 76E25, 76W05

Keywords: Hall-MHD, Global strong solutions, Existence, Stability

1 Introduction

This paper is concerned with the 3D incompressible Hall-MHD system

$$\left\{ \begin{array}{lll} \partial_t u + [u, \nabla]u + \nabla p - (\nabla \times b) \times b & = & \mu \Delta u \quad \text{in } (x, t) \in \mathbb{R}^3 \times [0, \infty); \\ \partial_t b - \nabla \times (u \times b) + \nabla \times ((\nabla \times b) \times b) & = & \gamma \Delta b \quad \text{in } (x, t) \in \mathbb{R}^3 \times [0, \infty); \\ \operatorname{div} u & = & 0 \quad \text{in } (x, t) \in \mathbb{R}^3 \times [0, \infty), \end{array} \right. \quad (1.1)$$

where $u = (u_1(x, t), u_2(x, t), u_3(x, t))$ is the velocity field, $p = p(x, t)$ is the scalar pressure field, $b = (b_1(x, t), b_2(x, t), b_3(x, t))$ is the magnetic field induced by the charged fluid, $\mu > 0$ and $\gamma > 0$ are respectively the viscosity and resistivity coefficients, $[u, \nabla] = \sum_{i=1}^3 u_i \partial_{x_i}$ and the symbol \times stands for the usual three-dimensional cross-product. The density of the fluid is assumed to be one by normalization.

The system (1.1) has been studied in the physics literature for decades (see e.g. [2], [27] and their references) and has application in a number of physical fields such as geo-dynamo [31], neutron stars [37] and magnetic reconnection in plasmas [20]. The reader is referred to [2] (see also [6]) for a deduction

*Universidade Estadual de Campinas, IMECC-Departamento de Matemática, CEP 13083-859, Campinas-SP, Brazil. Email: mbenvenuti@hotmail.com. MJB was supported by FAPESP, Brazil.

†Universidade Estadual de Campinas, IMECC-Departamento de Matemática, CEP 13083-859, Campinas-SP, Brazil. Email: lcff@ime.unicamp.br. LCFF was supported by FAPESP and CNPQ, Brazil. (corresponding author)

of (1.1) from two-fluids model, as well as from kinetic model, considering a generalized Ohm law. In comparison with the usual incompressible MHD system (see [36]), we have the new term $\nabla \times ((\nabla \times b) \times b)$ which is due to Hall effect and prevents straightforward adaptations from arguments used in the mathematical analysis of Navier-Stokes and related models.

Unlike MHD system that has an extensive variety of studies in classical subjects such as existence of solutions, regularity criteria and stability (see e.g. [14], [16], [19], [36], [39], [40] and references therein), the influence of the Hall term has been little explored on these topics. Indeed, Hall-MHD has appeared only recently in the mathematical literature and there are relatively a few works with this type of approach which are reviewed in what follows. In [2], by using Galerkin's method, global in time existence of weak solutions is proved in the periodic setting $L^2([0, 1]^3)$ for the resistive ($\gamma > 0$) and viscous case ($\mu > 0$). The uniqueness of weak solutions is still an open problem. Considering $\mu \geq 0$ and $\gamma > 0$, the authors of [8] obtained, via energy method, local-in-time well-posedness of strong solutions in $H^m(\mathbb{R}^3)$ with $m > \frac{5}{2}$ as well as global well-posedness under small conditions. They also showed blow-up criteria of first type for strong solutions and a Liouville theorem for smooth stationary solutions. The main point in [8] was the control of the Hall term via diffusion induced by the resistivity (see more details in the next paragraph). In [9] some blow-up criteria are studied and it is obtained a global well-posedness result for small initial data in terms of Besov norm which can be considered optimal in a suitable way that takes into account the scaling property for the system with null velocity. A subclass of global strong axisymmetric solutions was obtained in [17]. By employing Fourier splitting method, time-decay of Sobolev norms is showed in [10] for a class of weak solutions. A version of (1.1) with magnetic fractional diffusion $(-\Delta)^\alpha$ was considered in [12], where it was proved local well-posedness in Sobolev spaces for any $\alpha > \frac{2}{3}$ by using the smoothing effects of the dissipation and local bounds for the Sobolev norms through a multi-stage process. Regularity criteria for the density-dependent case is studied in [18]. In [11] it is shown that the non-resistive system ($\gamma = 0$) is not globally well-posed in any Sobolev space $H^m(\mathbb{R}^3)$ with $m > \frac{7}{2}$ in the sense that either it is locally ill-posed or it is locally well-posed but there exists an axisymmetric solution that loses the initial regularity in finite time.

Due to the spatial derivative of high-order in a nonlinear term, the Hall-MHD leads us to deal with higher regularity in the energy estimates (see e.g. (4.16)-(4.18) and (4.24)-(4.30) in Section 4), which introduces further difficulties in handling the system. For comparison, let us recall briefly about the issue of well-posedness for the incompressible Navier-Stokes and Euler equations: the local well-posedness with large data and the global one with small initial data in $H^1(\mathbb{R}^3)$ for the Navier-Stokes equations are obtained via an energy inequality where the nonlinearity, which is of first order $([u \cdot \nabla]u)$, is estimated by using Gagliardo-Nirenberg inequality and the diffusion controls the generated second-order derivative (see [38]). In the inviscid case, the local well-posedness is obtained only in $H^m(\mathbb{R}^3)$ with $m > \frac{5}{2}$ where the key inclusion $H^m(\mathbb{R}^3) \hookrightarrow W^{1,\infty}(\mathbb{R}^3)$ holds true (see [30]). In [8], Chae *et al.* mixed these two approaches to prove a local existence theorem in $H^m(\mathbb{R}^3)$, for $\gamma > 0$ and $m > \frac{5}{2}$. Another structural difference is that the second-order derivatives in the Hall term seem to obstruct the parabolic regularization effect exhibited for instance by MHD and Navier-Stokes systems.

Our first result improves (in the viscosity case) the one of [8] by proving local-in-time well-posedness of (1.1) in $H^2(\mathbb{R}^3)$, and global well-posedness for small H^2 -initial data (see Theorem 3.1). Here we use accurate energy estimates and also the particular structure of the Hall term.

Global existence of strong solutions of (1.1) for large initial data is still an open challenging problem. With respect to this matter, as far as we know, there are just the above mentioned class of $2\frac{1}{2}$ dimensional solutions of the form $(b, u) = (b(r, z)e^\theta, u(r, z)e^r + u(r, z)e^z)$ as proved in [17]. We observe that the two-dimensional symmetry is not tractable due to the fact that in this situation the Hall term has just

the third component nonzero. Despite the helical symmetry is conserved for the system, it is an open question to prove that they are global in time. Let us again make a comparison with the Navier-Stokes and other classical systems. For Navier-Stokes equations, there are global strong solutions in the two-dimensional case (see e.g. [38]), under the condition of axial symmetry without swirl [24], and in the presence of helical symmetry [29]. In [34], Ponce *et al.* proved that global solutions with a suitable property are stable in the sense that solutions close to them are global as well. Fortunately, symmetric and two-dimensional solutions satisfy the hypothesis required and this gives a class of global large solutions which are genuinely three-dimensional (although approximately symmetric or two-dimensional). Related results can be found in [4], [5], [13], [21], [22], [23], [32] and [35]. There are similar theorems for inhomogeneous Navier-Stokes equations [1, 7], Boussinesq system [26, 28] and MHD system [25].

In this paper we extend the stability result of [34] to the system (1.1) (see Theorem 3.2). Again, the main difficulty is the Hall term that requires estimates to deal with higher derivatives in the nonlinear term. Considering the global solutions obtained in [34] for Navier-Stokes equations, our stability result provides a class of global strong solutions (v, h) for (1.1) with large velocities and small initial magnetic fields (see Remark 3.3).

This paper is organized in the following way: in Section 2 we give some definitions, recall some basic inequalities and vector identities, and discuss the formulation of the problem as well as the notions of weak and strong solutions. Section 3 is devoted to state our results. In Section 4 we obtain key estimates to deal with the system. Finally, the results are proved in Section 5.

2 Preliminary

2.1 Basic definitions and inequalities

Let us start with some basic definitions in order to formulate the problem. We denote by $H^m(\mathbb{R}^3)$ and $W^{m,p}(\mathbb{R}^3)$ the usual three-dimensional vector Sobolev spaces (see [38]). The subscript σ in $H^m(\mathbb{R}^3)$ or in L^p means that the vector fields are divergence-free. The classical Helmholtz orthogonal projection \mathbb{P} onto the space of the solenoidal functions is denoted by

$$\mathbb{P} : L^2(\mathbb{R}^3) \longmapsto L^2_\sigma(\mathbb{R}^3).$$

We recall the following particular cases of the Gagliardo-Nirenberg inequality in \mathbb{R}^3 (see [33])

$$\left\{ \begin{array}{ll} \|f\|_{L^6(\mathbb{R}^3)} \leq C \|\nabla f\|_{L^2(\mathbb{R}^3)}, & \forall f \in H^1(\mathbb{R}^3), \\ \|f\|_{L^3(\mathbb{R}^3)} \leq C \|f\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}} \|\nabla f\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}}, & \forall f \in H^1(\mathbb{R}^3), \\ \|f\|_{L^\infty(\mathbb{R}^3)} \leq C \|\nabla f\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}} \|\nabla^2 f\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}}, & \forall f \in H^2(\mathbb{R}^3). \end{array} \right. \quad (2.1)$$

Also, we will use some equivalent seminorms and norms that can be obtained easily by Fourier transform. We have that

$$\left\{ \begin{array}{ll} \|\nabla^2 f\|_{L^2(\mathbb{R}^3)}^2 \cong \|\Delta f\|_{L^2(\mathbb{R}^3)}^2 = \|\nabla \times \nabla \times f\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla \operatorname{div} f\|_{L^2(\mathbb{R}^3)}^2 & \text{in } H^2(\mathbb{R}^3), \\ \|\nabla^3 f\|_{L^2(\mathbb{R}^3)}^2 \cong \|\operatorname{div} \Delta f\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla \times \Delta f\|_{L^2(\mathbb{R}^3)}^2 & \text{in } H^3(\mathbb{R}^3), \\ \|\nabla^3 f\|_{L^2(\mathbb{R}^3)}^2 \cong \|\nabla \times \Delta f\|_{L^2(\mathbb{R}^3)}^2 & \text{in } H^3_\sigma(\mathbb{R}^3), \\ \|\nabla^3 f\|_{L^2(\mathbb{R}^3)}^2 \cong \|\operatorname{div} \Delta f\|_{L^2(\mathbb{R}^3)}^2 & \text{in } H^3_p(\mathbb{R}^3), \end{array} \right. \quad (2.2)$$

and

$$\left\{ \begin{array}{l} \|f\|_{H^2(\mathbb{R}^3)}^2 \approx \|f\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla f\|_{L^2(\mathbb{R}^3)}^2 + \|\Delta f\|_{L^2(\mathbb{R}^3)}^2, \\ \|f\|_{H^3(\mathbb{R}^3)}^2 \approx \|f\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla f\|_{L^2(\mathbb{R}^3)}^2 + \|\Delta f\|_{L^2(\mathbb{R}^3)}^2 + \|\operatorname{div} \Delta f\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla \times \Delta f\|_{L^2(\mathbb{R}^3)}^2, \\ \|f\|_{H^3_p(\mathbb{R}^3)}^2 \approx \|f\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla f\|_{L^2(\mathbb{R}^3)}^2 + \|\Delta f\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla \times \Delta f\|_{L^2(\mathbb{R}^3)}^2, \\ \|f\|_{H^3_p(\mathbb{R}^3)}^2 \approx \|f\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla f\|_{L^2(\mathbb{R}^3)}^2 + \|\Delta f\|_{L^2(\mathbb{R}^3)}^2 + \|\operatorname{div} \Delta f\|_{L^2(\mathbb{R}^3)}^2. \end{array} \right. \quad (2.3)$$

2.2 Vector identities

Here we recall some vector equalities which will be useful in order to deal with the Hall term. We have (see [15])

$$\left\{ \begin{array}{l} \Delta A = \nabla \operatorname{div} A - \nabla \times \nabla \times A, \\ (\nabla \times A) \times B = [A, \nabla]B + [B, \nabla]A + A \times (\nabla \times B) - \nabla(A \cdot B), \\ \nabla \times (A \times B) = A(\operatorname{div} B) - B(\operatorname{div} A) + [B, \nabla]A - [A, \nabla]B, \end{array} \right.$$

from which we obtain

$$\begin{aligned} \nabla \times ((\nabla \times A) \times B) - (\nabla \times \nabla \times A) \times B &= (\nabla \times A)(\operatorname{div} B) - 2[(\nabla \times A), \nabla]B \\ &\quad - (\nabla \times A) \times (\nabla \times B) + \nabla((\nabla \times A) \cdot B) \end{aligned} \quad (2.4)$$

and

$$\begin{aligned} \nabla \times ((\nabla \times \nabla \times A) \times B) - (\nabla \times \nabla \times \nabla \times A) \times B &= (\nabla \times \nabla \times A)(\operatorname{div} B) \\ &\quad - 2[(\nabla \times \nabla \times A), \nabla]B \\ &\quad - (\nabla \times \nabla \times A) \times (\nabla \times B) \\ &\quad + \nabla((\nabla \times \nabla \times A) \cdot B). \end{aligned} \quad (2.5)$$

2.3 Weak and strong solutions

Consider the operators

$$A_1 : H^1_\sigma(\mathbb{R}^3) \longmapsto H^{-1}_\sigma(\mathbb{R}^3) \text{ defined by } \langle A_1[u], v \rangle = \mu \int_{\mathbb{R}^2} \nabla u \cdot \nabla v \, dx;$$

$$A_2 : H^1_\sigma(\mathbb{R}^3) \times H^1_\sigma(\mathbb{R}^3) \longmapsto H^{-1}_\sigma(\mathbb{R}^3) \text{ defined by } \langle A_2[u, h], v \rangle = \int_{\mathbb{R}^2} ([u, \nabla]h) \cdot v \, dx;$$

$$A_3 : H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3) \longmapsto H^{-1}_\sigma(\mathbb{R}^3) \text{ defined by } \langle A_3[b, h], v \rangle = - \int_{\mathbb{R}^2} ((\nabla \times b) \times h) \cdot v \, dx;$$

$$B_1 : H^1(\mathbb{R}^3) \longmapsto H^{-2}(\mathbb{R}^3) \text{ defined by } \langle B_1[b], w \rangle = \gamma \int_{\mathbb{R}^2} \nabla b \cdot \nabla w \, dx;$$

$$B_2 : H^1_\sigma(\mathbb{R}^3) \times H^1(\mathbb{R}^3) \longmapsto H^{-2}(\mathbb{R}^3) \text{ defined by } \langle B_2[u, b], w \rangle = - \int_{\mathbb{R}^2} (\nabla \times (u \times b)) \cdot w \, dx;$$

$$B_3 : H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3) \longmapsto H^{-2}(\mathbb{R}^3) \text{ defined by } \langle B_3[b, h], w \rangle = \int_{\mathbb{R}^2} ((\nabla \times b) \times h) \cdot (\nabla \times w) \, dx.$$

It is straightforward to prove by Gagliardo-Nirenberg type inequalities that these operators are well-defined and continuous. We consider the usual weak formulation for (1.1)

$$\begin{cases} \frac{d}{dt}u + A_1[u] + A_2[u, u] + A_3[b, b] &= 0 \text{ in } L^1((0, T), H_\sigma^{-1}(\mathbb{R}^3)); \\ \frac{d}{dt}b + B_1[b] + B_2[u, b] + B_3[b, b] &= 0 \text{ in } L^1((0, T), H^{-2}(\mathbb{R}^3)); \\ \mathbb{P}[u] &= u. \end{cases} \quad (2.6)$$

For $(u_0, b_0) \in L_\sigma^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ and $0 < T < \infty$, we say that (u, b) is a weak solution in $(0, T)$ for (1.1) with initial data (u_0, b_0) if

$$\begin{cases} u \in L^2((0, T); H_\sigma^1(\mathbb{R}^3)) \cap L^\infty((0, T); L_\sigma^2(\mathbb{R}^3)), \\ b \in L^2((0, T); H^1(\mathbb{R}^3)) \cap L^\infty((0, T); L^2(\mathbb{R}^3)), \\ \frac{d}{dt}u \in L^1((0, T), H_\sigma^{-1}(\mathbb{R}^3)), \\ \frac{d}{dt}b \in L^1((0, T), H^{-2}(\mathbb{R}^3)) \end{cases} \quad (2.7)$$

and (u, b) satisfies (2.6). In the case $(0, \infty)$ (global solutions), we assume that (u, b) satisfies (2.6) and (2.7) for all $0 < T < \infty$.

Remark 2.1. If (u, b) is a weak solution, then $(u, b) \in C_w([0, T], L^2(\mathbb{R}^3))$ and the initial data condition is satisfied in an appropriate sense of weak limit (see [38]).

Inspired on the classical mathematical literature, it is natural to consider class of solutions in spaces where energy estimates provide, at least, local well-posedness. For Navier-Stokes equations (and also MHD), the space $L^2((0, T); H^2) \cap L^\infty((0, T); H^1)$ is commonly used together with H^1 initial data. These solutions are strong in the sense that they have H^1 -continuous orbits (i.e., belong to $C([0, T], H^1)$) and satisfy their respective systems in L^2 for almost everywhere $t \in (0, T)$.

Due the second-order derivative in the non-linear part of (1.1), the above space is not appropriated to perform suitable energy estimates. However, using the special structure of Hall term $\nabla \times ((\nabla \times b) \times b)$, we will prove the local well-posedness in $L^2((0, T); H_\sigma^3 \times H^3) \cap L^\infty((0, T); H_\sigma^2 \times H^2)$ for H^2 initial data and these solutions have H^2 -continuous orbits and satisfy the system in L^2 , for almost everywhere $t \in (0, T)$. So, we establish the following definition.

Definition 2.2. (Strong solution) Let $(u_0, b_0) \in H_\sigma^2(\mathbb{R}^3) \times H^2(\mathbb{R}^3)$. For $0 < T < \infty$, we say that (u, b) is a strong solution in $(0, T)$ for (1.1) with initial data (u_0, b_0) if (u, b) verifies (2.6) and belongs to class

$$\begin{cases} u \in L^2((0, T); H_\sigma^3(\mathbb{R}^3)) \cap L^\infty((0, T); H_\sigma^2(\mathbb{R}^3)), \\ b \in L^2((0, T); H^3(\mathbb{R}^3)) \cap L^\infty((0, T); H^2(\mathbb{R}^3)). \end{cases} \quad (2.8)$$

In the case of $(0, \infty)$ (global solutions), we assume that (u, b) satisfies (2.6) and (2.8) for all $0 < T < \infty$.

Remark 2.3. Indeed, we are going to prove uniqueness of weak solutions in a class larger than (2.8), namely $L^4((0, T), H_\sigma^1(\mathbb{R}^3)) \times L^4((0, T), H^2(\mathbb{R}^3))$ (see Theorem 3.1).

Remark 2.4. It is straightforward to check that a strong solution of (1.1) satisfies

$$\frac{d}{dt}\Delta u \in L^2((0, T); H_\sigma^{-1}(\mathbb{R}^3)) \text{ and } \frac{d}{dt}\Delta b \in L^2((0, T); H^{-1}(\mathbb{R}^3)). \quad (2.9)$$

Therefore

$$u, b \in C([0, T], H^2(\mathbb{R}^3)), \quad (2.10)$$

$\langle \frac{d}{dt}\Delta u, \Delta u \rangle = \frac{1}{2} \frac{d}{dt} \|\Delta u\|_{L^2(\mathbb{R}^3)}^2$ and $\langle \frac{d}{dt}\Delta b, \Delta b \rangle = \frac{1}{2} \frac{d}{dt} \|\Delta b\|_{L^2(\mathbb{R}^3)}^2$ (see [38] for further details). The same is obviously true for spatial derivatives of lower order.

3 Results

In this section we state our results. We start with a result which improves the initial data regularity condition in [8] for local-in-time well-posedness.

3.1 Local-in-time well-posedness in H^2

Theorem 3.1. *Let $(u_0, b_0) \in H_\sigma^2(\mathbb{R}^3) \times H^2(\mathbb{R}^3)$. Then, there exist $T = T(\|u_0\|_{H_\sigma^2(\mathbb{R}^3)}, \|b_0\|_{H^2(\mathbb{R}^3)}) > 0$ and a strong solution (u, b) of (1.1) in $(0, T)$ with initial data (u_0, b_0) . This solution is the unique weak solution in $L^4((0, T), H_\sigma^1(\mathbb{R}^3)) \times L^4((0, T), H^2(\mathbb{R}^3))$. Furthermore, if $\|u_0\|_{H_\sigma^2(\mathbb{R}^3)}^2 + \|b_0\|_{H^2(\mathbb{R}^3)}^2$ is small enough, then the solution is global in time. Finally, if $T < \infty$ is the maximal existence time, then*

$$\int_0^T \left(\|\nabla u\|_{L^2(\mathbb{R}^3)}^4 + \|\nabla b\|_{L^2(\mathbb{R}^3)}^4 + \|\Delta b\|_{L^2(\mathbb{R}^3)}^4 \right) dt = \infty. \quad (3.1)$$

3.2 Global stability of large solutions

In the next theorem we obtain stability of large global strong solutions whose integral in (3.1) is finite with $T = \infty$. Notice that this condition is natural because we are dealing with global solutions.

Theorem 3.2. *Let (u, b) be a global strong solution of (1.1) with initial data $(u_0, b_0) \in H_\sigma^2(\mathbb{R}^3) \times H^2(\mathbb{R}^3)$ and satisfying*

$$\int_0^\infty \left(\|\nabla u\|_{L^2(\mathbb{R}^3)}^4 + \|\nabla b\|_{L^2(\mathbb{R}^3)}^4 + \|\Delta b\|_{L^2(\mathbb{R}^3)}^4 \right) dt < \infty. \quad (3.2)$$

There exists $\delta > 0$ such that if $(v_0, h_0) \in H_\sigma^2(\mathbb{R}^3) \times H^2(\mathbb{R}^3)$ and

$$\|u_0 - v_0\|_{H_\sigma^2(\mathbb{R}^3)}^2 + \|b_0 - h_0\|_{H^2(\mathbb{R}^3)}^2 < \delta, \quad (3.3)$$

then the strong solution (v, h) with initial data (v_0, h_0) is global in time. Furthermore, there exists $M = M(\delta)$ with $M(\delta) \xrightarrow{\delta \rightarrow 0} 0$ such that

$$\sup_{t \geq 0} \left(\|u(t) - v(t)\|_{H_\sigma^2(\mathbb{R}^3)}^2 + \|b(t) - h(t)\|_{H^2(\mathbb{R}^3)}^2 \right) \leq M(\delta).$$

Remark 3.3. *In Theorem 3.1, global strong solutions are obtained for small initial velocities and magnetic fields. We can use Theorem 3.2 to provide a class of global strong solutions with large initial velocities and small initial magnetic fields. Let us consider the classical incompressible Navier-Stokes equations*

$$\begin{cases} \partial_t u + [u, \nabla]u + \nabla p &= \mu \Delta u & \text{in } (x, t) \in \mathbb{R}^3 \times [0, \infty); \\ \operatorname{div} u &= 0 & \text{in } (x, t) \in \mathbb{R}^3 \times [0, \infty). \end{cases} \quad (3.4)$$

As pointed out in Introduction, the paper [34] provides a class of global large solutions for (3.4) satisfying

$$\int_0^\infty \|\nabla u(s)\|_{L^2(\mathbb{R}^3)}^4 ds < \infty. \quad (3.5)$$

We have that $(u, b) \equiv (u, 0)$ is a global strong solution for (1.1) and verifies (3.2). If (v, h) is a local-in-time strong solution for (1.1) (given by Theorem 3.1) such that $v(0)$ is close to $u(0)$ and $h(0)$ is small enough, then (v, h) is also a global strong solution.

4 Key estimates

We start with two lemmas which contain energy estimates that will be used to prove the results stated in Section 3. For the sake of presentation, the proof of Lemma 4.1 is postponed for Subsection 4.1.

Lemma 4.1. *Let (u, b) be a strong solution of (1.1) in $(0, T)$ according to Definition 2.2. We have that*

$$\frac{1}{2} \frac{d}{dt} (\|u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2) + \mu \|\nabla u(t)\|_{L^2}^2 + \gamma \|\nabla b(t)\|_{L^2}^2 = 0, \quad \forall 0 \leq t < T. \quad (4.1)$$

Furthermore, there are constants $C_0 = C_0(\mu, \gamma) > 0$ and $C_1 = C_1(\mu, \gamma) > 0$ such that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (\|\nabla u(t)\|_{L^2}^2 + \|\nabla b(t)\|_{L^2}^2) + \frac{\mu}{2} \|\Delta u(t)\|_{L^2}^2 + \frac{\gamma}{2} \|\Delta b(t)\|_{L^2}^2 \\ \leq C_0 (\|\nabla u(t)\|_{L^2}^2 + \|\nabla b(t)\|_{L^2}^2)^3 + \|\Delta b(t)\|_{L^2}^4 \|\nabla b(t)\|_{L^2}^2, \quad \forall 0 \leq t < T, \end{aligned} \quad (4.2)$$

and

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (\|\Delta u(t)\|_{L^2}^2 + \|\Delta b(t)\|_{L^2}^2) + \frac{\mu}{4} \|\nabla \times \Delta u(t)\|_{L^2}^2 + \frac{\gamma}{4} (\|\nabla \times \Delta b(t)\|_{L^2}^2 + \|\operatorname{div} \Delta b(t)\|_{L^2}^2) \\ \leq \frac{\gamma}{4} \|\Delta b(t)\|_{L^2}^2 + C_1 (\|\nabla u(t)\|_{L^2}^2 + \|\nabla b(t)\|_{L^2}^2 + \|\Delta b(t)\|_{L^2}^2)^3 \\ + C_1 \|\Delta u(t)\|_{L^2}^2 (\|\nabla u(t)\|_{L^2}^4 + \|\nabla b(t)\|_{L^2}^4), \quad \forall 0 \leq t < T. \end{aligned} \quad (4.3)$$

Remark 4.2. *Using (4.1)-(4.3) together with the equivalent norms (2.2)-(2.3), we have that the above solution satisfies*

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left(\|u(t)\|_{H^2(\mathbb{R}^3)}^2 + \|b(t)\|_{H^2(\mathbb{R}^3)}^2 \right) + \frac{\mu}{4} \left(\|\nabla u(t)\|_{L^2(\mathbb{R}^3)}^2 + \|\Delta u(t)\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla \times \Delta u(t)\|_{L^2(\mathbb{R}^3)}^2 \right) \\ + \frac{\gamma}{4} \left(\|\nabla b(t)\|_{L^2(\mathbb{R}^3)}^2 + \|\Delta b(t)\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla \times \Delta b(t)\|_{L^2(\mathbb{R}^3)}^2 + \|\operatorname{div} \Delta b(t)\|_{L^2(\mathbb{R}^3)}^2 \right) \\ \leq C \left(\|u(t)\|_{H^2(\mathbb{R}^3)}^2 + \|b(t)\|_{H^2(\mathbb{R}^3)}^2 \right) \left(\|\nabla u(t)\|_{L^2(\mathbb{R}^3)}^4 + \|\nabla b(t)\|_{L^2(\mathbb{R}^3)}^4 + \|\Delta b(t)\|_{L^2(\mathbb{R}^3)}^4 \right). \end{aligned} \quad (4.4)$$

The subject of the next lemma is to show that solutions as in Lemma 4.1 under the condition (3.2) satisfy a stronger estimate that gives some control on the Hall-term.

Lemma 4.3. *Let (u, b) be a global strong solution that satisfies (3.2). Then*

$$\int_0^\infty \|\nabla u(s)\|_{L^2}^4 + \|\nabla b(s)\|_{L^2}^4 + \|\Delta u(s)\|_{L^2}^4 + \|\Delta b(s)\|_{L^2}^4 + \|\nabla \times \Delta b(s)\|_{L^2}^2 + \|\operatorname{div} \Delta b(s)\|_{L^2}^2 ds < \infty. \quad (4.5)$$

Proof. Let $f(t) = \|u(t)\|_{H^2(\mathbb{R}^3)}^2 + \|b(t)\|_{H^2(\mathbb{R}^3)}^2$. So, by (4.4), we have

$$\frac{1}{2} \frac{d}{dt} f(t) \leq C f(t) \left(\|\nabla u(s)\|_{L^2(\mathbb{R}^3)}^4 + \|\nabla b(s)\|_{L^2(\mathbb{R}^3)}^4 + \|\Delta b(s)\|_{L^2(\mathbb{R}^3)}^4 \right). \quad (4.6)$$

If we define

$$M = \int_0^\infty \left(\|\nabla u(s)\|_{L^2(\mathbb{R}^3)}^4 + \|\nabla b(s)\|_{L^2(\mathbb{R}^3)}^4 + \|\Delta b(s)\|_{L^2(\mathbb{R}^3)}^4 \right) ds < \infty$$

and use Gronwall inequality in (4.6), we obtain

$$\|u(t)\|_{H^2(\mathbb{R}^3)}^2 + \|b(t)\|_{H^2(\mathbb{R}^3)}^2 \leq e^{2CM} \left(\|u(0)\|_{H^2(\mathbb{R}^3)}^2 + \|b(0)\|_{H^2(\mathbb{R}^3)}^2 \right). \quad (4.7)$$

On the other side, integrating (4.4), we also get

$$\begin{aligned} \frac{\gamma}{4} \int_0^t \left(\|\nabla \times \Delta b(s)\|_{L^2(\mathbb{R}^3)}^2 + \|\operatorname{div} \Delta b(s)\|_{L^2(\mathbb{R}^3)}^2 \right) ds + \frac{\mu}{4} \int_0^t \|\Delta u(s)\|_{L^2(\mathbb{R}^3)}^2 ds \\ \leq CM \sup_{s>0} \left\{ \|u(s)\|_{H^2(\mathbb{R}^3)}^2 + \|b(s)\|_{H^2(\mathbb{R}^3)}^2 \right\}. \end{aligned} \quad (4.8)$$

The condition (3.2) and inequalities (4.7)-(4.8) give (4.5). ■

In order to obtain the stability result, we need to estimate the difference between two strong solutions of (1.1). For this, we have two lemmas whose proofs are relatively long and so we also postpone them for later (see subsections 4.2 and 4.3).

Lemma 4.4. *Let (u, b) and (v, h) be strong solutions of (1.1). If $U = v - u$ and $B = h - b$ then*

$$\left\{ \begin{array}{l} \partial_t U - \mu \Delta U = -P[[U, \nabla]U] - P[[U, \nabla]u] - P[[u, \nabla]U] + P[(\nabla \times B) \times B] \\ \quad + P[(\nabla \times B) \times b] + P[(\nabla \times b) \times B]; \\ \partial_t B - \gamma \Delta B = \nabla \times (U \times B) + \nabla \times (U \times b) + \nabla \times (u \times B) - \nabla \times ((\nabla \times B) \times B) \\ \quad - \nabla \times ((\nabla \times B) \times b) - \nabla \times ((\nabla \times b) \times B); \\ \operatorname{div} U = 0. \end{array} \right. \quad (4.9)$$

Furthermore, there are constants $C_2 = C_2(\mu, \gamma) > 0$ and $C_3 = C_3(\mu, \gamma) > 0$ such that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (\|U(t)\|_{L^2}^2 + \|B(t)\|_{L^2}^2) + \frac{\mu}{2} \|\nabla U(t)\|_{L^2}^2 + \frac{\gamma}{2} \|\nabla B(t)\|_{L^2}^2 \\ \leq C_2 (\|\nabla u(t)\|_{L^2}^4 + \|\nabla b(t)\|_{L^2}^4 + \|\Delta b(t)\|_{L^2}^4) (\|U(t)\|_{L^2}^2 + \|B(t)\|_{L^2}^2), \quad \forall 0 \leq t < T, \end{aligned} \quad (4.10)$$

and

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (\|\nabla U(t)\|_{L^2}^2 + \|\nabla B(t)\|_{L^2}^2) + \frac{\mu}{2} \|\Delta U(t)\|_{L^2}^2 + \frac{\gamma}{2} \|\Delta B(t)\|_{L^2}^2 \\ \leq C_3 (\|\nabla U(t)\|_{L^2}^2 + \|\nabla B(t)\|_{L^2}^2 + \|\Delta B(t)\|_{L^2}^2) (\|\nabla u(t)\|_{L^2}^4 + \|\nabla b(t)\|_{L^2}^4 + \|\Delta b(t)\|_{L^2}^4) \\ + C_3 (\|\nabla U(t)\|_{L^2}^2 + \|\nabla B(t)\|_{L^2}^2 + \|\Delta B(t)\|_{L^2}^2)^3, \quad \forall 0 \leq t < T. \end{aligned} \quad (4.11)$$

Remark 4.5. *The inequality (4.10) holds true for two weak solutions (u, b) and (v, h) belonging to $L^4((0, T), H_\sigma^1(\mathbb{R}^3)) \times L^4((0, T), H^2(\mathbb{R}^3))$.*

Lemma 4.6. *Let (u, b) , (v, h) and (U, B) as in Lemma 4.4. There is a constant $C_4 = C_4(\mu, \gamma) > 0$ such that*

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|\Delta U\|_{L^2}^2 + \|\Delta B\|_{L^2}^2) + \frac{\mu}{4} \|\nabla \times \Delta U\|_{L^2}^2 + \frac{\gamma}{4} (\|\nabla \times \Delta B\|_{L^2}^2 + \|\operatorname{div} \Delta B\|_{L^2}^2) \\ & \leq C_4 (\|\nabla U\|_{L^2}^2 + \|\nabla B\|_{L^2}^2 + \|\Delta U\|_{L^2}^2 + \|\Delta B\|_{L^2}^2) (\|\nabla u\|_{L^2}^4 + \|\nabla b\|_{L^4}^4 + \|\Delta u\|_{L^2}^4 + \|\Delta b\|_{L^2}^4) \\ & + \frac{\mu}{4} \|\Delta U\|_{L^2}^2 + \frac{\gamma}{4} \|\Delta B\|_{L^2}^2 + C_4 (\|\nabla B\|_{L^2}^2 + \|\Delta B\|_{L^2}^2) (\|\nabla \times \Delta b\|_{L^2}^2 + \|\operatorname{div} \Delta b\|_{L^2}^2) \\ & + C_4 (\|\nabla U\|_{L^2}^2 + \|\nabla B\|_{L^2}^2 + \|\Delta U\|_{L^2}^2 + \|\Delta B\|_{L^2}^2)^3, \quad \forall 0 \leq t < T. \end{aligned} \quad (4.12)$$

Remark 4.7. *Let $U = v - u$, $B = h - b$ and*

$$L(t) = \|\nabla u\|_{L^2}^4 + \|\nabla b\|_{L^2}^4 + \|\Delta u\|_{L^2}^4 + \|\Delta b\|_{L^2}^4 + \|\nabla \times \Delta b\|_{L^2}^2 + \|\operatorname{div} \Delta b\|_{L^2}^2.$$

It follows from (4.10)-(4.12) that

$$\begin{aligned} & \frac{d}{dt} (\|U\|_{H^2}^2 + \|B\|_{H^2}^2) + \frac{\mu}{4} (\|\nabla U\|_{L^2(\mathbb{R}^3)}^2 + \|\Delta U\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla \times \Delta U\|_{L^2(\mathbb{R}^3)}^2) \\ & + \frac{\gamma}{4} (\|\nabla B\|_{L^2(\mathbb{R}^3)}^2 + \|\Delta B\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla \times \Delta B\|_{L^2(\mathbb{R}^3)}^2 + \|\operatorname{div} \Delta B\|_{L^2(\mathbb{R}^3)}^2) \\ & \leq C (\|U\|_{H^2}^2 + \|B\|_{H^2}^2)^3 + C (\|U\|_{H^2}^2 + \|B\|_{H^2}^2) L(t). \end{aligned}$$

4.1 Proof of Lemma 4.1

We multiply the first and second equations in (2.6) by u and b , respectively, and afterwards we integrate to obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|u\|_{L^2}^2 + \|b\|_{L^2}^2) + \mu \|\nabla u\|_{L^2}^2 + \gamma \|\nabla b\|_{L^2}^2 = - \overbrace{(P[[u.\nabla]u], u)_{L^2}}^{I_1} \\ & + \overbrace{(P[(\nabla \times b) \times b], u)_{L^2}}^{I_2} + \overbrace{(\nabla \times (u \times b), b)_{L^2}}^{I_3} \\ & - \overbrace{(\nabla \times ((\nabla \times b) \times b), b)_{L^2}}^{I_4}. \end{aligned} \quad (4.13)$$

Performing an integration by parts and using the incompressible condition to u , we get

$$I_1 = ([u.\nabla]u, u)_{L^2} = -([u.\nabla]u, u)_{L^2} = 0,$$

$$I_2 = ((\nabla \times b) \times b, u)_{L^2} = ([b.\nabla]b - \frac{1}{2}\nabla|b|^2, u)_{L^2} = ([b.\nabla]b, u)_{L^2},$$

$$\begin{aligned} I_3 &= (\nabla \times (u \times b), b)_{L^2} = (u(\operatorname{div} b) + [b.\nabla]u - [u.\nabla]b, b)_{L^2} = (u(\operatorname{div} b) + [b.\nabla]u, b)_{L^2} \\ &= -([b.\nabla]b, u)_{L^2} = -I_2, \end{aligned}$$

$$I_4 = ((\nabla \times b) \times b, \nabla \times b)_{L^2} = 0.$$

Inserting the above equalities in (4.13), we obtain (4.1).

Now, we multiply the first equation of (2.6) by $-\Delta u$ and the second by $-\Delta b$ in order to obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2) + \mu \|\Delta u\|_{L^2}^2 + \gamma \|\Delta b\|_{L^2}^2 &= \overbrace{(P[u \cdot \nabla]u, \Delta u)_{L^2}}^{J_1} \\ &\quad - \overbrace{(P[(\nabla \times b) \times b], \Delta u)_{L^2}}^{J_2} - \overbrace{(\nabla \times (u \times b), \Delta b)_{L^2}}^{J_3} \\ &\quad + \overbrace{(\nabla \times ((\nabla \times b) \times b), \Delta b)_{L^2}}^{J_4}. \end{aligned} \quad (4.14)$$

The terms J_i on the right side of (4.14) can be estimated by using Gagliardo-Nirenberg inequalities and Young inequality with ϵ . Hereafter C will be positive constants that can change in each line and they may depend of μ, γ and ϵ . Precisely, we have

$$\begin{aligned} |J_1| &= |(u \cdot \nabla)u, \Delta u|_{L^2} \leq C \|\Delta u\|_{L^2} \|\nabla u\|_{L^3} \|u\|_{L^6} \\ &\leq C \|\Delta u\|_{L^2} \left(\|\nabla u\|_{L^2}^{\frac{1}{2}} \|\Delta u\|_{L^2}^{\frac{1}{2}} \right) \|\nabla u\|_{L^2} \leq \epsilon \|\Delta u\|_{L^2}^2 + C \|\nabla u\|_{L^2}^6, \\ |J_2| &= |((\nabla \times b) \times b, \Delta u)_{L^2}| \leq C \|\Delta u\|_{L^2} \|\nabla b\|_{L^3} \|b\|_{L^6} \\ &\leq C \|\Delta u\|_{L^2} \left(\|\nabla b\|_{L^2}^{\frac{1}{2}} \|\Delta b\|_{L^2}^{\frac{1}{2}} \right) \|\nabla b\|_{L^2} \leq \epsilon \|\Delta u\|_{L^2}^2 + \epsilon \|\Delta b\|_{L^2}^2 + C \|\nabla b\|_{L^2}^6, \\ |J_3| &= |(\nabla \times (u \times b), \Delta b)_{L^2}| \leq C \|\Delta b\|_{L^2} (\|\nabla b\|_{L^3} \|u\|_{L^6} + \|\nabla u\|_{L^2} \|b\|_{L^\infty}) \\ &\leq C \|\Delta b\|_{L^2} \left(\|\nabla b\|_{L^2}^{\frac{1}{2}} \|\Delta b\|_{L^2}^{\frac{1}{2}} \right) \|\nabla u\|_{L^2} \leq \epsilon \|\Delta u\|_{L^2}^2 + C \|\nabla u\|_{L^2}^4 \|\nabla b\|_{L^2}^2, \\ |J_4| &= |(\nabla \times ((\nabla \times b) \times b), \Delta b)_{L^2}| \leq C \|\Delta b\|_{L^2} (\|\nabla b\|_{L^3} \|\nabla b\|_{L^6} + \|b\|_{L^\infty} \|\Delta b\|_{L^2}) \\ &\leq C \|\Delta b\|_{L^2} \left(\|\nabla b\|_{L^2}^{\frac{1}{2}} \|\Delta b\|_{L^2}^{\frac{1}{2}} \right) \|\Delta b\|_{L^2} \leq \epsilon \|\Delta b\|_{L^2}^2 + C \|\Delta b\|_{L^2}^4 \|\nabla b\|_{L^2}^2. \end{aligned}$$

The inequality (4.2) follows by inserting the above estimates in (4.14) and choosing $\epsilon > 0$ small enough.

Finally, we deal with (4.3). After applying $\nabla \times$ in (2.6), we multiply the first equation by $-\nabla \times \Delta u$ and the second by $-\nabla \times \Delta b$. Also we apply div in the second equation of (2.6) and then multiply it by $-\operatorname{div} \Delta b$. With these manipulations, we get the following equality

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (\|\Delta u\|_{L^2}^2 + \|\Delta b\|_{L^2}^2) + \mu \|\nabla \times \Delta u\|_{L^2}^2 + \gamma (\|\nabla \times \Delta b\|_{L^2}^2 + \|\operatorname{div} \Delta b\|_{L^2}^2) \\ = \overbrace{(\nabla \times ([u \cdot \nabla]u), \nabla \times \Delta u)_{L^2}}^{K_1} - \overbrace{(\nabla \times ((\nabla \times b) \times b), \nabla \times \Delta u)_{L^2}}^{K_2} \\ - \overbrace{(\nabla \times \nabla \times (u \times b), \nabla \times \Delta b)_{L^2}}^{K_3} + \overbrace{(\nabla \times \nabla \times ((\nabla \times b) \times b), \nabla \times \Delta b)_{L^2}}^{K_4}. \end{aligned} \quad (4.15)$$

For the first three terms on the right side of (4.15), we have

$$\begin{aligned}
|K_1| &\leq C\|\nabla \times \Delta u\|_{L^2} (\|u\|_{L^6}\|\Delta u\|_{L^3} + \|\nabla u\|_{L^\infty}\|\nabla u\|_{L^2}) \\
&\leq C\|\nabla \times \Delta u\|_{L^2} \left(\|\nabla \times \Delta u\|_{L^2}^{\frac{1}{2}} \|\Delta u\|_{L^2}^{\frac{1}{2}} \right) \|\nabla u\|_{L^2} \\
&\leq \epsilon \|\nabla \times \Delta u\|_{L^2}^2 + C\|\Delta u\|_{L^2}^2 \|\nabla u\|_{L^2}^4, \\
|K_2| &\leq C\|\nabla \times \Delta u\|_{L^2} (\|b\|_{L^\infty}\|\Delta b\|_{L^2} + \|\nabla b\|_{L^3}\|\nabla b\|_{L^6}) \\
&\leq C\|\nabla \times \Delta u\|_{L^2} \left(\|\nabla b\|_{L^2}^{\frac{1}{2}} \|\Delta b\|_{L^2}^{\frac{1}{2}} \right) \|\Delta b\|_{L^2} \\
&\leq \epsilon \|\nabla \times \Delta u\|_{L^2}^2 + \epsilon \|\Delta b\|_{L^2}^2 + C\|\Delta b\|_{L^2}^4 \|\nabla b\|_{L^2}^2,
\end{aligned}$$

and

$$\begin{aligned}
|K_3| &\leq C\|\nabla \times \Delta b\|_{L^2} (\|u\|_{L^6}\|\Delta b\|_{L^3} + \|\nabla u\|_{L^2}\|\nabla b\|_{L^\infty}) \\
&\quad + C\|\nabla \times \Delta b\|_{L^2} (\|b\|_{L^6}\|\Delta u\|_{L^3}) \\
&\leq C\|\nabla \times \Delta b\|_{L^2} \left(\|\Delta b\|_{L^2}^{\frac{1}{2}} \left(\|\nabla \times \Delta b\|_{L^2}^{\frac{1}{2}} + \|\operatorname{div} \Delta b\|_{L^2}^{\frac{1}{2}} \right) \right) \|\nabla u\|_{L^2} \\
&\quad + C\|\nabla \times \Delta b\|_{L^2} \left(\|\nabla \times \Delta u\|_{L^2}^{\frac{1}{2}} \|\Delta u\|_{L^2}^{\frac{1}{2}} \right) \|\nabla b\|_{L^2} \\
&\leq \epsilon \|\nabla \times \Delta u\|_{L^2}^2 + \epsilon \|\nabla \times \Delta b\|_{L^2}^2 + \epsilon \|\operatorname{div} \Delta b\|_{L^2}^2 + C\|\Delta b\|_{L^2}^2 \|\nabla u\|_{L^2}^4 + C\|\Delta u\|_{L^2}^2 \|\nabla b\|_{L^2}^4.
\end{aligned}$$

For the Hall-term, using the vector identity $(A \times B) \cdot A = 0$, we get

$$\begin{aligned}
K_4 &= (\nabla \times \nabla \times ((\nabla \times b) \times b), -\nabla \times \nabla \times \nabla \times b)_{L^2} \\
&= \overbrace{(\nabla \times \nabla \times ((\nabla \times b) \times b) - \nabla \times ((\nabla \times \nabla \times b) \times b), -\nabla \times \nabla \times \nabla \times b)}^{K_5}_{L^2} \\
&\quad + \overbrace{(\nabla \times ((\nabla \times \nabla \times b) \times b) - (\nabla \times \nabla \times \nabla \times b) \times b, -\nabla \times \nabla \times \nabla \times b)}^{K_6}_{L^2}. \tag{4.16}
\end{aligned}$$

Now, by identities (2.4)-(2.5), we obtain

$$\begin{aligned}
|K_5| &= |(\nabla \times \{\nabla \times ((\nabla \times b) \times b) - (\nabla \times \nabla \times b) \times b\}, \nabla \times \Delta b)_{L^2}| \\
&= |(\nabla \times \{(\nabla \times b)(\operatorname{div} b) - 2[(\nabla \times b) \cdot \nabla]b\}, \nabla \times \Delta b)_{L^2}| \\
&\leq C\|\nabla \times \Delta b\|_{L^2} \|\nabla b\|_{L^6} \|\Delta b\|_{L^3} \\
&\leq C\|\nabla \times \Delta b\|_{L^2} \|\Delta b\|_{L^2} \left(\|\Delta b\|_{L^2}^{\frac{1}{2}} \left(\|\nabla \times \Delta b\|_{L^2}^{\frac{1}{2}} + \|\operatorname{div} \Delta b\|_{L^2}^{\frac{1}{2}} \right) \right) \\
&\leq \epsilon \|\nabla \times \Delta b\|_{L^2}^2 + \epsilon \|\operatorname{div} \Delta b\|_{L^2}^2 + C\|\Delta b\|_{L^2}^6 \tag{4.17}
\end{aligned}$$

and

$$\begin{aligned}
|K_6| &= |(\nabla \times \nabla \times b)(\operatorname{div} b) - 2[(\nabla \times \nabla \times b) \cdot \nabla]b - (\nabla \times \nabla \times b) \times (\nabla \times b), \nabla \times \Delta b)_{L^2}| \\
&\leq C\|\nabla \times \Delta b\|_{L^2} \|\Delta b\|_{L^3} \|\nabla b\|_{L^6} \\
&\leq C\|\nabla \times \Delta b\|_{L^2} \left(\|\Delta b\|_{L^2}^{\frac{1}{2}} \left(\|\nabla \times \Delta b\|_{L^2}^{\frac{1}{2}} + \|\operatorname{div} \Delta b\|_{L^2}^{\frac{1}{2}} \right) \right) \|\Delta b\|_{L^2} \\
&\leq \epsilon \|\nabla \times \Delta b\|_{L^2}^2 + \epsilon \|\operatorname{div} \Delta b\|_{L^2}^2 + C\|\Delta b\|_{L^2}^6. \tag{4.18}
\end{aligned}$$

We obtain (4.3) from (4.15), the above estimates for $|K_i|$, and choosing a suitable $\epsilon > 0$ small enough. ■

4.2 Proof of Lemma 4.4

The proof of (4.9) is a straightforward calculation. Now we multiply the first equation of (4.9) by U and the second equation by B to obtain

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} (\|U\|_{L^2}^2 + \|B\|_{L^2}^2) + \mu \|\nabla U\|_{L^2}^2 + \gamma \|\nabla B\|_{L^2}^2 = & - \overbrace{(P[[U.\nabla]U], U)}^{L_1}_{L^2} - \overbrace{(P[[U.\nabla]u], U)}^{L_2}_{L^2} \\
& - \overbrace{(P[[u.\nabla]U], U)}^{L_3}_{L^2} + \overbrace{(P[(\nabla \times B) \times B], U)}^{L_4}_{L^2} + \overbrace{(P[(\nabla \times B) \times b], U)}^{L_5}_{L^2} \\
& + \overbrace{(P[(\nabla \times b) \times B], U)}^{L_6}_{L^2} + \overbrace{(\nabla \times (U \times B), B)}^{L_7}_{L^2} + \overbrace{(\nabla \times (U \times b), B)}^{L_8}_{L^2} \\
& + \overbrace{(\nabla \times (u \times B), B)}^{L_9}_{L^2} - \overbrace{(\nabla \times ((\nabla \times B) \times B), B)}^{L_{10}}_{L^2} - \overbrace{(\nabla \times ((\nabla \times B) \times b), B)}^{L_{11}}_{L^2} \\
& - \overbrace{(\nabla \times ((\nabla \times b) \times B), B)}^{L_{12}}_{L^2}. \tag{4.19}
\end{aligned}$$

We have that $L_1 = L_3 = L_{10} = L_{11} = 0$ and $L_4 = -L_7$. Then, we need to estimate the remainder terms. Proceeding as in the proof of Lemma 4.1, we obtain

$$\begin{aligned}
|L_2| & \leq \epsilon \|\nabla U\|_{L^2}^2 + C \|\nabla u\|_{L^2}^4 \|U\|_{L^2}^2, \\
|L_5| & \leq \epsilon \|\nabla B\|_{L^2}^2 + C \|\nabla b\|_{L^2}^4 \|B\|_{L^2}^2, \\
|L_6| & \leq \epsilon \|\nabla B\|_{L^2}^2 + \epsilon \|\nabla U\|_{L^2}^2 + C \|\nabla b\|_{L^2}^4 \|B\|_{L^2}^2, \\
|L_8| & \leq \epsilon \|\nabla B\|_{L^2}^2 + \epsilon \|\nabla U\|_{L^2}^2 + C \|\nabla b\|_{L^2}^4 \|B\|_{L^2}^2, \\
|L_9| & \leq \epsilon \|\nabla B\|_{L^2}^2 + C \|\nabla u\|_{L^2}^4 \|B\|_{L^2}^2, \\
|L_{12}| & \leq \epsilon \|\nabla B\|_{L^2}^2 + C \|\Delta b\|_{L^2}^4 \|B\|_{L^2}^2.
\end{aligned}$$

Now we obtain (4.10) after inserting the above inequalities in (4.19) and taking $\epsilon > 0$ small enough.

Next we prove (4.11). Multiplying the first equation of (4.9) by $-\Delta U$ and the second by $-\Delta B$, we get

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} (\|\nabla U\|_{L^2}^2 + \|\nabla B\|_{L^2}^2) + \mu \|\Delta U\|_{L^2}^2 + \gamma \|\Delta B\|_{L^2}^2 = & \overbrace{(P[[U.\nabla]U], \Delta U)}^{M_1}_{L^2} + \overbrace{(P[[U.\nabla]u], \Delta U)}^{M_2}_{L^2} \\
& + \overbrace{(P[[u.\nabla]U], \Delta U)}^{M_3}_{L^2} - \overbrace{(P[(\nabla \times B) \times B], \Delta U)}^{M_4}_{L^2} - \overbrace{(P[(\nabla \times B) \times b], \Delta U)}^{M_5}_{L^2} \\
& - \overbrace{(P[(\nabla \times b) \times B], \Delta U)}^{M_6}_{L^2} - \overbrace{(\nabla \times (U \times B), \Delta B)}^{M_7}_{L^2} - \overbrace{(\nabla \times (U \times b), \Delta B)}^{M_8}_{L^2} \\
& - \overbrace{(\nabla \times (u \times B), \Delta B)}^{M_9}_{L^2} + \overbrace{(\nabla \times ((\nabla \times B) \times B), \Delta B)}^{M_{10}}_{L^2} \\
& + \overbrace{(\nabla \times ((\nabla \times B) \times b), \Delta B)}^{M_{11}}_{L^2} + \overbrace{(\nabla \times ((\nabla \times b) \times B), \Delta B)}^{M_{12}}_{L^2}. \tag{4.20}
\end{aligned}$$

We have the following estimates:

$$\begin{aligned}
|M_1| &\leq \epsilon \|\Delta U\|_{L^2}^2 + C \|\nabla U\|_{L^2}^6, \\
|M_2| &\leq \epsilon \|\Delta U\|_{L^2}^2 + C \|\nabla u\|_{L^2}^4 \|\nabla U\|_{L^2}^2, \\
|M_3| &\leq \epsilon \|\Delta U\|_{L^2}^2 + C \|\nabla u\|_{L^2}^4 \|\nabla U\|_{L^2}^2, \\
|M_4| &\leq \epsilon \|\Delta U\|_{L^2}^2 + \epsilon \|\Delta B\|_{L^2}^2 + C \|\nabla B\|_{L^2}^6, \\
|M_5| &\leq \epsilon \|\Delta U\|_{L^2}^2 + \epsilon \|\Delta B\|_{L^2}^2 + C \|\nabla b\|_{L^2}^4 \|\nabla B\|_{L^2}^2, \\
|M_6| &\leq \epsilon \|\Delta U\|_{L^2}^2 + \epsilon \|\Delta B\|_{L^2}^2 + C \|\nabla b\|_{L^2}^4 \|\nabla B\|_{L^2}^2, \\
|M_7| &\leq \epsilon \|\Delta U\|_{L^2}^2 + \epsilon \|\Delta B\|_{L^2}^2 + C \|\nabla B\|_{L^2}^4 \|\nabla U\|_{L^2}^2, \\
|M_8| &\leq \epsilon \|\Delta U\|_{L^2}^2 + \epsilon \|\Delta B\|_{L^2}^2 + C \|\nabla b\|_{L^2}^4 \|\nabla U\|_{L^2}^2, \\
|M_9| &\leq \epsilon \|\Delta B\|_{L^2}^2 + C \|\nabla u\|_{L^2}^4 \|\nabla B\|_{L^2}^2, \\
|M_{10}| &\leq \epsilon \|\Delta B\|_{L^2}^2 + C \|\Delta B\|_{L^2}^4 \|\nabla B\|_{L^2}^2,
\end{aligned}$$

and

$$\begin{aligned}
|M_{11}| &\leq \epsilon \|\Delta B\|_{L^2}^2 + C \|\Delta b\|_{L^2}^4 \|\nabla B\|_{L^2}^2 + C \|\Delta B\|_{L^2}^2 \|\nabla b\|_{L^2}^4 + C \|\Delta B\|_{L^2}^2 \|\Delta b\|_{L^2}^4, \\
|M_{12}| &\leq \epsilon \|\Delta B\|_{L^2}^2 + C \|\Delta b\|_{L^2}^4 \|\nabla B\|_{L^2}^2.
\end{aligned}$$

Again, taking $\epsilon > 0$ small enough, the above estimates together with (4.20) give (4.11). ■

4.3 Proof of Lemma 4.6

First we apply $\nabla \times$ in (4.9) and then multiply the first equation by $-\nabla \times \Delta U$ and the second equation by $-\nabla \times \Delta B$. Also, we apply div in the second equation of (4.9) and we multiply it by $-\operatorname{div} \Delta B$. After this manipulations, we obtain

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} (\|\Delta U\|_{L^2}^2 + \|\Delta B\|_{L^2}^2) + \mu \|\nabla \times \Delta U\|_{L^2}^2 + \gamma (\|\nabla \times \Delta B\|_{L^2}^2 + \|\operatorname{div} \Delta B\|_{L^2}^2) \\
&= \overbrace{(\nabla \times ([U \cdot \nabla] U), \nabla \times \Delta U)_{L^2}}^{N_1} + \overbrace{(\nabla \times ([U \cdot \nabla] u), \nabla \times \Delta U)_{L^2}}^{N_2} \\
&+ \overbrace{(\nabla \times ([u \cdot \nabla] U), \nabla \times \Delta U)_{L^2}}^{N_3} - \overbrace{(\nabla \times ((\nabla \times B) \times B), \nabla \times \Delta U)_{L^2}}^{N_4} \\
&- \overbrace{(\nabla \times ((\nabla \times B) \times b), \nabla \times \Delta U)_{L^2}}^{N_5} - \overbrace{(\nabla \times ((\nabla \times b) \times B), \nabla \times \Delta U)_{L^2}}^{N_6} \\
&- \overbrace{(\nabla \times \nabla \times (U \times B), \nabla \times \Delta B)_{L^2}}^{N_7} - \overbrace{(\nabla \times \nabla \times (U \times b), \nabla \times \Delta B)_{L^2}}^{N_8} \\
&- \overbrace{(\nabla \times \nabla \times (u \times B), \nabla \times \Delta B)_{L^2}}^{N_9} + \overbrace{(\nabla \times \nabla \times ((\nabla \times B) \times B), \nabla \times \Delta B)_{L^2}}^{N_{10}} \\
&+ \overbrace{(\nabla \times \nabla \times ((\nabla \times B) \times b), \nabla \times \Delta B)_{L^2}}^{N_{11}} + \overbrace{(\nabla \times \nabla \times ((\nabla \times b) \times B), \nabla \times \Delta B)_{L^2}}^{N_{12}}.
\end{aligned} \tag{4.21}$$

As before, we can estimate

$$\begin{aligned}
|N_1| &\leq \epsilon \|\nabla \times \Delta U\|_{L^2}^2 + \epsilon \|\Delta U\|_{L^2}^2 + C \|\Delta U\|_{L^2}^4 \|\nabla U\|_{L^2}^2, \\
|N_2| &\leq \epsilon \|\nabla \times \Delta U\|_{L^2}^2 + \epsilon \|\Delta U\|_{L^2}^2 + C \|\Delta u\|_{L^2}^4 \|\nabla U\|_{L^2}^2, \\
|N_3| &\leq \epsilon \|\nabla \times \Delta U\|_{L^2}^2 + C \|\nabla u\|_{L^2}^4 \|\Delta U\|_{L^2}^2, \\
|N_4| &\leq \epsilon \|\nabla \times \Delta U\|_{L^2}^2 + \epsilon \|\Delta B\|_{L^2}^2 + C \|\nabla B\|_{L^2}^2 \|\Delta B\|_{L^2}^4, \\
|N_5| &\leq \epsilon \|\nabla \times \Delta U\|_{L^2}^2 + \epsilon \|\nabla \times \Delta B\|_{L^2}^2 + \epsilon \|\operatorname{div} \Delta B\|_{L^2}^2 + C \|\nabla b\|_{L^2}^4 \|\Delta B\|_{L^2}^2,
\end{aligned} \tag{4.22}$$

and

$$\begin{aligned}
|N_6| &\leq \epsilon \|\nabla \times \Delta U\|_{L^2}^2 + \epsilon \|\Delta B\|_{L^2}^2 + C \|\Delta b\|_{L^2}^4 \|\nabla B\|_{L^2}^2, \\
|N_7| &\leq \epsilon \|\Delta U\|_{L^2}^2 + \epsilon \|\Delta B\|_{L^2}^2 + \epsilon \|\nabla \times \Delta B\|_{L^2}^2 + C \|\Delta B\|_{L^2}^4 \|\nabla U\|_{L^2}^2 + C \|\Delta U\|_{L^2}^4 \|\nabla B\|_{L^2}^2, \\
|N_8| &\leq \epsilon \|\nabla \times \Delta B\|_{L^2}^2 + \epsilon \|\nabla \times \Delta U\|_{L^2}^2 + \epsilon \|\Delta U\|_{L^2}^2 + C \|\nabla b\|_{L^2}^4 \|\Delta U\|_{L^2}^2 + C \|\nabla U\|_{L^2}^2 \|\Delta b\|_{L^2}^4, \\
|N_9| &\leq \epsilon \|\nabla \times \Delta B\|_{L^2}^2 + \epsilon \|\operatorname{div} \Delta B\|_{L^2}^2 + \epsilon \|\Delta B\|_{L^2}^2 + C \|\nabla u\|_{L^2}^4 \|\Delta B\|_{L^2}^2 + C \|\nabla B\|_{L^2}^2 \|\Delta u\|_{L^2}^4.
\end{aligned} \tag{4.23}$$

For the Hall-term, using the vector identity $(A \times B) \cdot A = 0$, we get

$$\begin{aligned}
N_{10} &= (\nabla \times \nabla \times ((\nabla \times B) \times B), -\nabla \times \nabla \times \nabla \times B)_{L^2} \\
&= \overbrace{(\nabla \times \nabla \times ((\nabla \times B) \times B) - \nabla \times ((\nabla \times \nabla \times B) \times B), -\nabla \times \nabla \times \nabla \times B)}^{N_{10,a}}_{L^2} \\
&+ \overbrace{(\nabla \times ((\nabla \times \nabla \times B) \times B) - (\nabla \times \nabla \times \nabla \times B) \times B, -\nabla \times \nabla \times \nabla \times B)}^{N_{10,b}}_{L^2} \tag{4.24}
\end{aligned}$$

and

$$\begin{aligned}
N_{11} &= (\nabla \times \nabla \times ((\nabla \times B) \times b), -\nabla \times \nabla \times \nabla \times B)_{L^2} \\
&= \overbrace{(\nabla \times \nabla \times ((\nabla \times B) \times b) - \nabla \times ((\nabla \times \nabla \times B) \times b), -\nabla \times \nabla \times \nabla \times B)}^{N_{11,a}}_{L^2} \\
&+ \overbrace{(\nabla \times ((\nabla \times \nabla \times B) \times b) - (\nabla \times \nabla \times \nabla \times B) \times b, -\nabla \times \nabla \times \nabla \times B)}^{N_{11,b}}_{L^2}. \tag{4.25}
\end{aligned}$$

Now, by using identities (2.4)-(2.5), we obtain

$$\begin{aligned}
|N_{10,a}| &= |(\nabla \times \{\nabla \times ((\nabla \times B) \times B) - (\nabla \times \nabla \times B) \times B\}, \nabla \times \Delta B)_{L^2}| \\
&= |(\nabla \times \{(\nabla \times B)(\operatorname{div} B) - 2[(\nabla \times B) \cdot \nabla] B\}, \nabla \times \Delta B)_{L^2}| \\
&\leq C \|\Delta B\|_{L^2} \|\nabla B\|_{L^\infty} \|\nabla \times \Delta B\|_{L^2} \\
&\leq C \|\Delta B\|_{L^2}^{\frac{3}{2}} \left(\|\nabla \times \Delta B\|_{L^2}^{\frac{1}{2}} + \|\operatorname{div} \Delta B\|_{L^2}^{\frac{1}{2}} \right) \|\nabla \times \Delta B\|_{L^2} \\
&\leq \epsilon \|\nabla \times \Delta B\|_{L^2}^2 + \epsilon \|\operatorname{div} \Delta B\|_{L^2}^2 + C \|\Delta B\|_{L^2}^6,
\end{aligned} \tag{4.26}$$

$$\begin{aligned}
|N_{10,b}| &= |((\nabla \times \nabla \times B)(\operatorname{div} B) - 2[(\nabla \times \nabla \times B) \cdot \nabla] B - (\nabla \times \nabla \times B) \times (\nabla \times B), \nabla \times \Delta B)_{L^2}| \\
&\leq C \|\Delta B\|_{L^2} \|\nabla B\|_{L^\infty} \|\nabla \times \Delta B\|_{L^2} \\
&\leq C \|\Delta B\|_{L^2}^{\frac{3}{2}} \left(\|\nabla \times \Delta B\|_{L^2}^{\frac{1}{2}} + \|\operatorname{div} \Delta B\|_{L^2}^{\frac{1}{2}} \right) \|\nabla \times \Delta B\|_{L^2} \\
&\leq \epsilon \|\nabla \times \Delta B\|_{L^2}^2 + \epsilon \|\operatorname{div} \Delta B\|_{L^2}^2 + C \|\Delta B\|_{L^2}^6,
\end{aligned} \tag{4.27}$$

$$\begin{aligned}
|N_{11,a}| &= |(\nabla \times \{(\nabla \times B)(\operatorname{div} b) - 2[(\nabla \times B) \cdot \nabla]b - (\nabla \times B) \times (\nabla \times b)\}, \nabla \times \Delta B)_{L^2}| \\
&\leq C(\|\Delta B\|_{L^3}\|\nabla b\|_{L^6} + \|\Delta b\|_{L^2}\|\nabla B\|_{L^\infty})\|\nabla \times \Delta B\|_{L^2} \\
&\leq C\|\Delta b\|_{L^2}\|\Delta B\|_{L^2}^{\frac{1}{2}}\left(\|\nabla \times \Delta B\|_{L^2}^{\frac{1}{2}} + \|\operatorname{div} \Delta B\|_{L^2}^{\frac{1}{2}}\right)\|\nabla \times \Delta B\|_{L^2} \\
&\leq \epsilon\|\nabla \times \Delta B\|_{L^2}^2 + \epsilon\|\operatorname{div} \Delta B\|_{L^2}^2 + C\|\Delta B\|_{L^2}^2\|\Delta b\|_{L^2}^4,
\end{aligned} \tag{4.28}$$

$$\begin{aligned}
|N_{11,b}| &= |((\nabla \times \nabla \times B)(\operatorname{div} b) - 2[(\nabla \times \nabla \times B) \cdot \nabla]b - (\nabla \times \nabla \times B) \times (\nabla \times b), \nabla \times \Delta B)_{L^2}| \\
&\leq C\|\Delta B\|_{L^3}\|\nabla b\|_{L^6}\|\nabla \times \Delta B\|_{L^2} \\
&\leq C\|\Delta B\|_{L^2}^{\frac{1}{2}}\left(\|\nabla \times \Delta B\|_{L^2}^{\frac{1}{2}} + \|\operatorname{div} \Delta B\|_{L^2}^{\frac{1}{2}}\right)\|\Delta b\|_{L^2}\|\nabla \times \Delta B\|_{L^2} \\
&\leq \epsilon\|\nabla \times \Delta B\|_{L^2}^2 + \epsilon\|\operatorname{div} \Delta B\|_{L^2}^2 + C\|\Delta B\|_{L^2}^2\|\Delta b\|_{L^2}^4,
\end{aligned} \tag{4.29}$$

and

$$\begin{aligned}
|N_{12}| &= |(\nabla \times \nabla \times ((\nabla \times b) \times B), \nabla \times \Delta B)_{L^2}| \\
&\leq C((\|\nabla \times \Delta b\|_{L^2} + \|\operatorname{div} \Delta b\|_{L^2})\|B\|_{L^\infty} + \|\Delta b\|_{L^6}\|\nabla B\|_{L^3})\|\nabla \times \Delta B\|_{L^2} \\
&\quad + \|\Delta B\|_{L^3}\|\nabla b\|_{L^6}\|\nabla \times \Delta B\|_{L^2} \\
&\leq C\left((\|\nabla \times \Delta b\|_{L^2} + \|\operatorname{div} \Delta b\|_{L^2})\|\nabla B\|_{L^2}^{\frac{1}{2}}\|\Delta B\|_{L^2}^{\frac{1}{2}}\right)\|\nabla \times \Delta B\|_{L^2} \\
&\quad + \|\Delta B\|_{L^2}^{\frac{1}{2}}\left(\|\nabla \times \Delta B\|_{L^2}^{\frac{1}{2}} + \|\operatorname{div} \Delta B\|_{L^2}^{\frac{1}{2}}\right)\|\Delta b\|_{L^2}\|\nabla \times \Delta B\|_{L^2} \\
&\leq \epsilon\|\nabla \times \Delta B\|_{L^2}^2 + \epsilon\|\operatorname{div} \Delta B\|_{L^2}^2 + C\|\Delta B\|_{L^2}^2\|\Delta b\|_{L^2}^4 \\
&\quad + C(\|\nabla B\|_{L^2}^2 + \|\Delta B\|_{L^2}^2)(\|\nabla \times \Delta b\|_{L^2}^2 + \|\operatorname{div} \Delta b\|_{L^2}^2).
\end{aligned} \tag{4.30}$$

We conclude the proof of (4.12) by considering the estimates (4.22)-(4.30) in (4.21) and taking a suitable $\epsilon > 0$.

■

5 Proof of Results

5.1 Proof of Theorem 3.1

Let $\rho \in C_0^\infty(\mathbb{R}^3)$ be a non-negative radial scalar function with unit integral and $0 \leq \rho \leq 1$ and let $\mathcal{J}_n \equiv \rho_n \ast$ be the standard mollifier, where $\rho_n(x) = n^3 \rho(nx)$ and $n \in \mathbb{N}$ (see [3]). We consider the regularized system

$$\left\{ \begin{aligned} \partial_t u + \mathcal{J}_n \mathbb{P}[(\mathcal{J}_n u \cdot \nabla) \mathcal{J}_n u - (\nabla \times \mathcal{J}_n b) \times \mathcal{J}_n b] &= \mu \Delta \mathcal{J}_n^2 u; \\ \partial_t b - \nabla \times \mathcal{J}_n (\mathcal{J}_n u \times \mathcal{J}_n b) + \nabla \times \mathcal{J}_n ((\nabla \times \mathcal{J}_n b) \times \mathcal{J}_n b) &= \gamma \Delta \mathcal{J}_n^2 b; \\ \mathbb{P}[u] &= u; \\ (u_{n,0}, b_{n,0}) &= (\mathcal{J}_n u_0, \mathcal{J}_n b_0). \end{aligned} \right. \tag{5.1}$$

In [8, Proposition 3.1] (see also [30]), global-in-time existence of smooth solutions for (5.1) is obtained for each fixed $n \in \mathbb{N}$. More precisely, the above system is considered as an autonomous infinite-dimensional ODE system in $H_\sigma^m \times H^m$ and it is used the generalized Picard theorem in Banach spaces. The key point is the suitable way that the original system is mollified by \mathcal{J}_n which implies simpler estimates for (5.1). For further details see [8] and [30].

Let (u_n, b_n) be a global-in-time smooth solution of (5.1) with initial data $(u_{n,0}, b_{n,0})$. Adapting for (5.1) the *a priori* estimates contained in Lemma 4.1 and Remark 4.2, we obtain a constant $C > 0$ (independent of n) such that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left(\|u_n\|_{H^2(\mathbb{R}^3)}^2 + \|b_n\|_{H^2(\mathbb{R}^3)}^2 \right) + \frac{\mu}{4} \left(\|\nabla \mathcal{J}_n u_n\|_{L^2(\mathbb{R}^3)}^2 + \|\Delta \mathcal{J}_n u_n\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla \times \Delta \mathcal{J}_n u_n\|_{L^2(\mathbb{R}^3)}^2 \right) \\ & + \frac{\gamma}{4} \left(\|\nabla \mathcal{J}_n b_n\|_{L^2(\mathbb{R}^3)}^2 + \|\Delta \mathcal{J}_n b_n\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla \times \Delta \mathcal{J}_n b_n\|_{L^2(\mathbb{R}^3)}^2 + \|\operatorname{div} \Delta \mathcal{J}_n b_n\|_{L^2(\mathbb{R}^3)}^2 \right) \\ & \leq C \left(\|u_n\|_{H^2(\mathbb{R}^3)}^2 + \|b_n\|_{H^2(\mathbb{R}^3)}^2 \right) \left(\|\nabla u_n\|_{L^2(\mathbb{R}^3)}^4 + \|\nabla b_n\|_{L^2(\mathbb{R}^3)}^4 + \|\Delta b_n\|_{L^2(\mathbb{R}^3)}^4 \right). \end{aligned} \quad (5.2)$$

It follows that

$$\frac{1}{2} \frac{d}{dt} \left(\|u_n(t)\|_{H^2(\mathbb{R}^3)}^2 + \|b_n(t)\|_{H^2(\mathbb{R}^3)}^2 \right) \leq C \left(\|u_n(t)\|_{H^2(\mathbb{R}^3)}^2 + \|b_n(t)\|_{H^2(\mathbb{R}^3)}^2 \right)^3.$$

Solving the differential inequality $\frac{d}{dt} f_n \leq C f_n^3$, with $f_n(t) = \|u_n(t)\|_{H^2(\mathbb{R}^3)}^2 + \|b_n(t)\|_{H^2(\mathbb{R}^3)}^2$, we get

$$f_n^2(t) \leq \frac{f_n^2(0)}{1 - 2Ct f_n^2(0)}.$$

Then, using the boundedness of $(f_n(0))_{n \in \mathbb{N}}$ by $f_0 = \|u_0\|_{H^2(\mathbb{R}^3)}^2 + \|b_0\|_{H^2(\mathbb{R}^3)}^2$ and fixing $0 < T^* < \frac{1}{2Cf_0^2}$, we obtain that

$$(u_n)_{n \in \mathbb{N}} \text{ and } (b_n)_{n \in \mathbb{N}} \text{ are bounded in } L^\infty((0, T^*), H^2(\mathbb{R}^3)). \quad (5.3)$$

Using again (5.2) and equations in (5.1), we can prove that

$$(\mathcal{J}_n u_n)_{n \in \mathbb{N}} \text{ and } (\mathcal{J}_n b_n)_{n \in \mathbb{N}} \text{ are bounded in } L^2((0, T^*), H^3(\mathbb{R}^3)), \quad (5.4)$$

$$(\partial_t u_n)_{n \in \mathbb{N}} \text{ and } (\partial_t b_n)_{n \in \mathbb{N}} \text{ are bounded in } L^\infty((0, T^*), L^2(\mathbb{R}^3)). \quad (5.5)$$

By (5.3)-(5.5), there exist a sub-sequence of $(u_n, b_n)_{n \in \mathbb{N}}$ (still indexed by n) and functions $u, b \in L^\infty((0, T^*), H^2(\mathbb{R}^3)) \cap L^2((0, T^*), H^3(\mathbb{R}^3))$ such that (see [38])

$$\begin{aligned} & (u_n, b_n) \xrightarrow{n \rightarrow \infty} (u, b) \text{ weak-* in } L^\infty((0, T^*), H_\sigma^2(\mathbb{R}^3) \times H^2(\mathbb{R}^3)), \\ & (\mathcal{J}_n u_n, \mathcal{J}_n b_n) \xrightarrow{n \rightarrow \infty} (u, b) \text{ weak in } L^2((0, T^*), H_\sigma^3(\mathbb{R}^3) \times H^3(\mathbb{R}^3)), \\ & (u_n, b_n) \xrightarrow{n \rightarrow \infty} (u, b) \text{ strong in } L^2((0, T^*), L_{loc}^2(\mathbb{R}^3) \times L_{loc}^2(\mathbb{R}^3)), \\ & (\partial_t u_n, \partial_t b_n) \xrightarrow{n \rightarrow \infty} (u, b) \text{ weak-* in } L^\infty((0, T^*), L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3)). \end{aligned}$$

With these properties, we can apply the weak limit in (5.1) and prove that (u, b) is a local strong solution of (1.1) with initial data (u_0, b_0) .

Let us to prove the uniqueness. Suppose that (u, b) and (v, h) are two weak solutions of (1.1) with the same initial data. Let $U = v - u$ and $B = h - b$. So, by inequality (4.10) (see Remark 4.5), we have that

$$\frac{1}{2} \frac{d}{dt} (\|U(t)\|_{L^2}^2 + \|B(t)\|_{L^2}^2) \leq C_2 (\|\nabla u(t)\|_{L^2}^4 + \|\nabla b(t)\|_{L^2}^4 + \|\Delta b(t)\|_{L^2}^4) (\|U(t)\|_{L^2}^2 + \|B(t)\|_{L^2}^2). \quad (5.6)$$

Now the uniqueness of solutions in $L^4((0, T^*), H_\sigma^1(\mathbb{R}^3) \times H^2(\mathbb{R}^3))$ follows from (5.6) and Gronwall inequality.

In what follows, we prove the blow-up criterion (3.1). If fact, suppose that (u, b) satisfies

$$M := \int_0^T (\|\nabla u(s)\|_{L^2(\mathbb{R}^3)}^4 + \|\nabla b(s)\|_{L^2(\mathbb{R}^3)}^4 + \|\Delta b(s)\|_{L^2(\mathbb{R}^3)}^4) ds < \infty.$$

Proceeding in an analogous way to the proof of Lemma 4.3, we obtain

$$\|u(t)\|_{H^2(\mathbb{R}^3)}^2 + \|b(t)\|_{H^2(\mathbb{R}^3)}^2 \leq e^{2CM} (\|u(0)\|_{H^2(\mathbb{R}^3)}^2 + \|b(0)\|_{H^2(\mathbb{R}^3)}^2).$$

So, by using the usual blow-up criterion of time-continuous H^2 -solutions, we have that the solution can be extended beyond T (see [9] and [38]).

Finally, let us prove global solutions for small initial data. By inequality (4.4) and equivalences (2.2)-(2.3), there exist $C_5 > 0$ and $C_6 > 0$ such that

$$\begin{aligned} \frac{d}{dt} (\|b(t)\|_{H^2}^2 + \|u(t)\|_{H^2}^2) &+ C_5 (\|u(t)\|_{H^2}^2 + \|b(t)\|_{H^2}^2) \\ &\leq C_6 (\|b(t)\|_{H^2}^2 + \|u(t)\|_{H^2}^2)^3 + C_5 (\|u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2). \end{aligned} \quad (5.7)$$

Suppose that the initial data is small enough to satisfy

$$\|u(0)\|_{H^2}^2 + \|b(0)\|_{H^2}^2 \leq \frac{1}{12} \sqrt{\frac{C_5}{C_6}}.$$

Let $T^* > 0$ be the supremum over all finite $\tilde{T} > 0$ such that

$$\|u(t)\|_{H^2}^2 + \|b(t)\|_{H^2}^2 \leq \sqrt{\frac{C_5}{2C_6}}, \quad \forall 0 \leq t \leq \tilde{T}.$$

By contradiction, let us assume that $0 < T^* < \infty$. By (5.7) we get

$$\frac{d}{dt} (\|b(t)\|_{H^2}^2 + \|u(t)\|_{H^2}^2) + \frac{C_5}{2} (\|u(t)\|_{H^2}^2 + \|b(t)\|_{H^2}^2) \leq C_5 (\|u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2), \quad \forall 0 \leq t \leq \tilde{T},$$

for all $0 < \tilde{T} < T^*$. Then, Gronwall type inequality and the time-uniform boundedness $\|u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2 \leq \|u(0)\|_{L^2}^2 + \|b(0)\|_{L^2}^2$ give

$$\begin{aligned} \|b(t)\|_{H^2}^2 + \|u(t)\|_{H^2}^2 &\leq e^{-\frac{C_5 t}{2}} \left(\|u(0)\|_{H^2}^2 + \|b(0)\|_{H^2}^2 + \int_0^t e^{\frac{C_5 s}{2}} C_5 (\|u(s)\|_{L^2}^2 + \|b(s)\|_{L^2}^2) ds \right) \\ &\leq 3(\|u(0)\|_{H^2}^2 + \|b(0)\|_{H^2}^2) \leq \frac{1}{4} \sqrt{\frac{C_5}{C_6}}, \quad \forall 0 \leq t \leq \tilde{T}, \end{aligned} \quad (5.8)$$

for all $0 < \tilde{T} < T^*$. In view of the time-continuity of (u, b) (see Remark 2.4) and $\frac{1}{4} \sqrt{\frac{C_5}{C_6}} < \sqrt{\frac{C_5}{2C_6}}$, the estimate (5.8) contradicts the maximality of T^* . So, $T^* = \infty$ and the solution is global in time. ■

5.2 Proof of Theorem 3.2

Let $U = v - u$, $B = h - b$ and

$$L(t) = \|\nabla u\|_{L^2}^4 + \|\nabla b\|_{L^2}^4 + \|\Delta u\|_{L^2}^4 + \|\Delta b\|_{L^2}^4 + \|\nabla \times \Delta b\|_{L^2}^2 + \|\operatorname{div} \Delta b\|_{L^2}^2.$$

By Remark 4.7, there are $C_{13} > 0$, $C_{14} > 0$ and $C_{15} > 0$ such that

$$\begin{aligned} \frac{d}{dt} (\|U\|_{H^2}^2 + \|B\|_{H^2}^2) + C_{13} (\|U\|_{H^2}^2 + \|B\|_{H^2}^2) &\leq C_{14} (\|U\|_{H^2}^2 + \|B\|_{H^2}^2)^3 \\ &+ C_{15} (\|U\|_{H^2}^2 + \|B\|_{H^2}^2) L(t) + C_{13} (\|U\|_{L^2}^2 + \|B\|_{L^2}^2). \end{aligned} \quad (5.9)$$

On the other side, using Lemma 4.3, we have that if (3.2) holds, then

$$\int_0^\infty L(s) ds < \infty.$$

Suppose that the initial data $(v(0), h(0))$ is close to $(u(0), b(0))$ to satisfy (here $C_2 > 0$ is given in (4.10))

$$\|U(0)\|_{H^2}^2 + \|B(0)\|_{H^2}^2 \leq \frac{1}{12} \sqrt{\frac{C_{13}}{C_{14}}} \frac{1}{e^{(C_{15}+2C_2) \int_0^\infty L(s) ds}}. \quad (5.10)$$

Let $T^* > 0$ be the supremum over all finite $\tilde{T} > 0$ such that

$$\|U(t)\|_{H^2}^2 + \|B(t)\|_{H^2}^2 \leq \sqrt{\frac{C_{13}}{2C_{14}}}, \quad \forall 0 \leq t \leq \tilde{T}.$$

Assume by contradiction that $T^* < \infty$. So, for all $0 < \tilde{T} < T^*$, we get

$$\begin{aligned} \frac{d}{dt} (\|U\|_{H^2}^2 + \|B\|_{H^2}^2) + \frac{C_{13}}{2} (\|U\|_{H^2}^2 + \|B\|_{H^2}^2) &\leq C_{15} (\|U\|_{H^2}^2 + \|B\|_{H^2}^2) L(t) \\ &+ C_{13} (\|U\|_{L^2}^2 + \|B\|_{L^2}^2) \quad \forall 0 \leq t \leq \tilde{T}. \end{aligned}$$

By Gronwall type inequality, for all $0 < \tilde{T} < T^*$, we obtain

$$\begin{aligned} \|U(t)\|_{H^2}^2 + \|B(t)\|_{H^2}^2 &\leq e^{-\frac{C_{13}}{2}t + C_{15} \int_0^t L(s) ds} (\|U(0)\|_{H^2}^2 + \|B(0)\|_{H^2}^2) \\ &+ e^{-\frac{C_{13}}{2}t + C_{15} \int_0^t L(s) ds} C_{13} \int_0^t e^{\frac{C_{13}}{2}s} (\|U(s)\|_{L^2}^2 + \|B(s)\|_{L^2}^2) ds \\ &\leq \frac{1}{12} \sqrt{\frac{C_{13}}{C_{14}}} + 2e^{C_{15} \int_0^\infty L(s) ds} \sup_{s>0} \{\|U(s)\|_{L^2}^2 + \|B(s)\|_{L^2}^2\}, \quad \forall 0 \leq t \leq \tilde{T}. \end{aligned}$$

Finally, it follows from (4.10) and Gronwall inequality that

$$\sup_{s>0} \{\|U(s)\|_{L^2}^2 + \|B(s)\|_{L^2}^2\} \leq e^{2C_2 \int_0^\infty L(s) ds} (\|U(0)\|_{L^2}^2 + \|B(0)\|_{L^2}^2), \quad \forall 0 \leq t \leq \tilde{T}.$$

So

$$\|U(t)\|_{H^2}^2 + \|B(t)\|_{H^2}^2 \leq \frac{1}{4} \sqrt{\frac{C_{13}}{C_{14}}}, \quad \forall 0 \leq t \leq \tilde{T}, \quad (5.11)$$

for all $0 < \tilde{T} < T^*$. The estimate (5.11) contradicts the maximality of T^* because $\frac{1}{4}\sqrt{\frac{C_{13}}{C_{14}}} < \sqrt{\frac{C_{13}}{2C_{14}}}$ and (U, B) is time-continuous (see Remark 2.4). It follows that $T^* = \infty$. As (u, b) is a global solution in $H^2(\mathbb{R}^3)$, the above inequality implies that (v, h) is as well. Furthermore, by repeating steps between (5.10)-(5.11), one can check that for initial data less than δ , where $0 < \delta < \frac{1}{12}\sqrt{\frac{C_{13}}{C_{14}}}\frac{1}{e^{(C_{15}+2C_2)\int_0^\infty L(s)ds}}$, we can take $M(\delta) = 3\delta e^{(C_{15}+2C_2)\int_0^\infty L(s)ds}$, for $M(\delta)$ as in the statement of the theorem. This concludes the proof. ■

References

- [1] ABIDI H., GUI G. & ZHANG P., *On the decay and stability of global solutions to the 3D inhomogeneous Navier-Stokes equations*, Comm. Pure App. Math. 64 (2011), 832-881.
- [2] ACHERITOGARAY M., DEGOND P., FROUVILLE A. & LIU J-G., *Kinetic formulation and global existence for the Hall-Magneto-hydrodynamics system*, Kinet. Relat. Models 4 (2011), 901-918.
- [3] ADAMS R.A. & FOURNIER, J.J.F., *Sobolev Spaces*, 2nd ed., Pure and Applied Mathematics 140, Elsevier/Academic Press, Amsterdam, 2003.
- [4] AUSCHER P., DUBOIS S. & TCHAMITCHIAN P., *On the stability of global solutions to Navier-Stokes equations in the space*, J. Math. Pures Appl. 83 (2004), 673-697.
- [5] BARDOS C., LOPES FILHO M. C., NIU D., NUSSENZVEIG LOPES, H.J. & TITI E., *Stability of two-dimensional viscous incompressible flows under three-dimensional perturbations and inviscid symmetry breaking*, SIAM J. Math. Anal. 45 (2013), 1871-1885.
- [6] BITTENCOURT J.A., *Fundamentals of plasma Physics*, Pergamon Press, New York, NY, 1986.
- [7] CAI X. J., JIU Q. S. & ZHOU Y. L., *Global L^2 of the nonhomogeneous Incompressible Navier-Stokes equations*, Acta Math. Sinica 29 (11) (2013), 2087-2098.
- [8] CHAE D., DEGOND P. & LIU J., *Well-posedness for Hall-magnetohydrodynamics*, Ann. Inst. H. Poincaré Anal. Non Linéaire 31 (2014), 555-565.
- [9] CHAE D. & LEE J., *On the blow-up criterion and small data global existence for the Hall-magnetohydrodynamics*, J. Diff. Eqs 256 (11) (2014), 3835-3858.
- [10] CHAE D. & SCHONBEK M., *On the temporal decay for the Hall-magnetohydrodynamics*, J. Diff. Eqs. 255 (11) (2013), 3971-3982.
- [11] CHAE D. & WENG S., *Singularity formation for the incompressible Hall-MHD equations without Resistivity*, arXiv: 1312.5519 (2013).
- [12] CHAE D. & WU J., *Local Well-posedness for the Hall-MHD equations with Fractional magnetic diffusion*, arXiv: 1404.0486 (2014).

- [13] CHEMIN J.-Y. & GALLAGHER I., *On the global wellposedness of the 3-D Navier-Stokes equations with Large initial data*, Ann. Sci. École Norm. Sup. 39 (4) (2006), 679-698.
- [14] CHEN Q., MIAO C. & ZHANG Z., *The Beale-Kato-Majda Criterion for the 3D Magneto-Hydrodynamics Equations*, Comm. Math. Phys. 275 (2007), 861-872.
- [15] CHORIN J. & MARSDEN J., *A mathematical introduction to fluid mechanics*, Texts in Applied Mathematics 4, Springer-Verlag, New York, NY, 2000.
- [16] DUVAUT G. & LIONS J.L., *Inéquations en thermoélasticité et magnétohydrodynamique*. (French) Arch. Rational Mech. Anal. 46 (1972), 241-279.
- [17] FAN J., HUANG S. & NAKAMURA G., *Well-posedness for the axisymmetric incompressible viscous Hall-magnetohydrodynamics equations*, Applied Mathematics Letters 26 (2013), 963-967.
- [18] FAN J. & OZAWA, *Regularity criteria for the density-dependent Hall-magnetohydrodynamics*, Applied Mathematics Letters 36 (2014), 14-18.
- [19] FERREIRA L.C.F. & VILLAMIZAR-ROA E.J., *Exponentially-stable steady flow and asymptotic behavior for the magnetohydrodynamic equations*, Comm. Math. Sci. 9 (2) (2011), 499-516.
- [20] FORBES T.G., *Magnetic reconnection in solar flares*, Geophys. Astrophys. Fluid Dynamics 62 (1-4) (1991), 15-36.
- [21] GALLAGHER I., IFTIMIE D. & PLANCHON F., *Asymptotics and stability for global solutions to the Navier-Stokes equations*, Ann. Inst. Fourier 53 (2003), 1387-1424.
- [22] GUI G. & ZHANG P., *Stability to the global large solutions of the 3-D Navier-Stokes equations*, Adv. Math. 225 (2010), 1248-1284.
- [23] IFTIMIE D., *The 3D Navier-Stokes equations seen as a perturbation of the 2D Navier-Stokes equations*, Bull. Soc. Math. France 127 (1999), 473-517.
- [24] LADYZHANSKAYA O.A., *Unique global solvability of the three-dimensional Cauchy problem for the Navier-Stokes equations in the presence of axial symmetry*, Zap. Nau. Sem. Leningrad. Otdel. Mat. Inst. Steklov 7 (1968), 155-177.
- [25] LI X. & CAI X., *The global L^2 stability of solutions to three dimensional MHD equations*, Acta Math. Scientia 33B (1) (2013), 247-267.
- [26] LI X. & JIU Q. S., *The global L^2 stability of Large solutions to three dimensional Boussinesq equations*, Acta Math. Sinica 53 (2010), 171-186.
- [27] LIGHTHILL M.J., *Studies on magneto-hydrodynamics waves and other anisotropic wave motion*, Philo. Trans. R. Soc. Lond. Ser A 252 (1960), 397-430.
- [28] LIU X. & LI Y., *On the stability of global solutions to the 3d Boussinesq system*, Nonlinear Analysis 95 (2014), 580-591.

- [29] MAHALOV A., TITI E. & LEIBOVICH S., *Invariant Helical subspace for the Navier-Stokes equations*, Arch. Rational Mech. Anal. 112 (1990), 193-222.
- [30] MAJDA A., BERTOZZI A., *Vorticity and incompressible flow*, Cambridge Texts in Applied Mathematics 27, Cambridge University Press, Cambridge, UK, 2001.
- [31] MININNI P.D., GOMEZ D.O. & MAHAJAN S.M., *Dynamo Action in magnetohydrodynamics and Hall magnetohydrodynamics*, Astrophys. J. 587 (2003), 472-481.
- [32] MUCHA P.B., *Stability of 2D incompressible flows in \mathbb{R}^3* , J. Diff. Eqs. (245) (9) (2008), 2355-2367.
- [33] NIRENBERG L., *On elliptic partial differential equations*, Ann. Scuola Norm. Sup. Pisa (3) (1959), 115-162.
- [34] PONCE G., RACKE R., SIDERIS T.C. & TITI E., *Global Stability of Large Solutions to the 3D Navier-Stokes Equations*, Comm. Math. Phys. 159 (1994), 329-341.
- [35] RUSIN W., *Navier-Stokes equations, Stability and minimal perturbations of global solutions*, J. Math. Anal. Appl. 386 (1) (2012), 115-124.
- [36] SERMANGE M. & TEMAM R., *Some mathematical questions related to the MHD equations*, Comm. Pure Appl. Math. 36 (1983), 635-664.
- [37] SHALYBKOV D.A. & URPIN V.A., *The Hall effect and the decay of magnetic field*, Astron. Astrophys 321 (1997), 685-690.
- [38] TEMAM R., *Navier-Stokes Equations: Theory and Numerical Analysis*, AMS Chelsea Publishing, Providence, RI, 2001.
- [39] WU J., *Regularity Criteria for the Generalized MHD Equations*, Comm. Part. Diff. Eqs. 33 (2008), 285-306.
- [40] ZHOU Y., *Regularity criteria for the generalized viscous MHD equations*, Ann. Inst. H. Poincaré Anal. Non Linéaire 24 (2007), 491-505.